Topological Quantum Computing Report I
Mathematical and Physical Background

Rachel Castro†

***Abstract*—Fault-tolerant quantum computing has been the focus of research since the model was first proposed in the 80s. One promising method is topological quantum computation, a rich and complex field. This paper discusses the physical background and mathematical foundations of topological quantum computation, with the aim of preparing the reader for the discussion of a topical paper in the field.**

I. Introduction and Motivation

In order to prepare students to understand a follow-on presentation, I provide the basic physical and mathematical background needed to understand topological quantum computing. In this paper, we discuss how phase acquisition can be path-dependent, using the Aharonov-Bohm effect to illustrate. Then we discuss how spin describes phase generation relative to rotations of space and why spin in two dimensions is unique. Finally, we describe the braid group B_n and its relation to fluxons.

II. Aharonov Bohm Effect

Students familiar with a beginner electromagnetic physics may recall that a charged particle q moved through a magnetic field $\vec{B}$ experiences a generated force proportional to the strength of the field and the charge of the particle:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

However, when the magnetic field is removed, it is clear the experienced force goes to zero. So a beginner student would be justified in assuming that a lack of magnetic field where the particle is traveling means there is no effect on the particle.

Thus consider the following situation, as depicted in the figure.

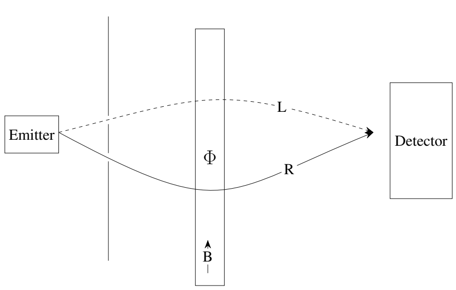


Fig. 1. The Aharonov-Bohm effect.

There is a magnetic field $\vec{B}$ enclosed in a flux tube such that there is no field outside of the tube. A particle is emitted from the source region A and proceeds through a double slit around the flux tube, with a left path L and right path R. The particle ends at B, where it is measured by a detector to be different than the standard double-slit experiment. What is causing this discrepancy?

The answer is one of the most fundamental phenomena in quantum mechanics: destructive interference [1]. By traversing the space around the flux tube in two different paths, the wave function of the particle acquired phases that cancel each other out, resulting in destructive interference. But there is no magnetic field outside of the flux tube, something else must be influencing this phase change. In fact, it is the potential field $\vec{A}$, where $\nabla \times \vec{A} = \vec{B}$ that affects the phase change. We have from electrodynamics that the phase $e^{i\phi}$ acquired by a particle with charge q on path P through a potential field $\vec{A}$ is the following:

$$e^{i\phi} = \frac{q}{\hbar} \int_P \vec{A} \cdot dl$$

The total phase acquired by the particle is path-dependent. If the wave function of the upper path is $e^{i\phi_R} |\psi\rangle$ and the lower path is $e^{i\phi_L} |\psi\rangle$, then the intensity at the detector is the difference between their phases:

$$e^{i\Delta\phi} = e^{i\phi_R} - e^{i\phi_L}$$
$$= \frac{q}{\hbar} \int_R \vec{A} \cdot dl - \frac{q}{\hbar} \int_L \vec{A} \cdot dl$$

Because the paths form a closed loop C, we can write:

$$e^{i\Delta\phi} = \frac{q}{\hbar} \oint_C \vec{A} \cdot dl$$

Then by Stokes' theorem, we have:

$$e^{i\Delta\phi} = \frac{q}{\hbar} \int \int_S (\nabla \times \vec{A}) \cdot da$$

Apply $\nabla \times \vec{A} = \vec{B}$ to obtain

$$e^{i\Delta\phi} = \frac{q}{\hbar} \int \int_S \vec{B} \cdot da$$

Where we know that the magnetic field over a surface is simply the flux of that surface. Then

$$e^{i\Delta\phi} = \Phi \frac{q}{\hbar}$$

which states that the amount of phase acquired is determined by the amount of flux enclosed in a path. (Calculation from [2]) Crucially, this means that the direction and number of times traveled around the flux tube impacts the phase generated, but the specific path itself does not. This fact is called **homotopy invariance**, which means that paths that can be smoothly deformed into each other are equivalent. This

†*Department of Mathematics, Tufts University*

is a form of topological equivalence that is fundamentally important to topological quantum computing. The acquired phase is often called **topological phase.** [3]

It is important to see that we can also generalize this notion acquired phase being dictated by rotation. For example, allow the flux tube to instead be a "fluxon"–a particle which contains some amount of magnetic field but does not allow the magnetic field to be interacted with. Then taking another charged particle around the fluxon has exactly the same effect as wrapping it around the tube. [1]

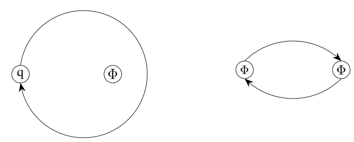


Fig. 2. Exchanges are equivalent to loops.

We can extend this once more to see that, instead of wrapping one particle around the other, if we simply allow them to swap places, it is exactly the same rotation relative to each other and thus exactly the same phase is acquired. We will assume for the rest of this report that we are discussing phase generation in the context of swapping particles. [1]

III. Physical Spin

In the modern theory of quantum mechanics, point-like particles are endowed with inherent properties: for example, mass and charge. One property is the inherent spin of a particle, which in three dimensions is either an integer or a half integer. In some texts, spin is described as a type of angular momentum. To support this, students are shown the Stern-Gerlach experiment, which shows how "spin up" and "spin down" particles are sorted by the orientation of bipolar magnets. In this report, we will characterize spin as the manner in which the waveform of a particle interacts with rotations of space. Let $|\psi\rangle$ be the wavefunction of a particle. Then if s is the spin of a particle, under a 2π rotation of space about an arbitrary axis, the particle acquires a phase:

$$|\psi\rangle \mapsto e^{i2\pi s} |\psi\rangle$$

[1] For example, in classic quantum mechanics we can consider bosons, which have any integer spin. Then for any 2π rotation of space, $|\psi\rangle \mapsto |\psi\rangle$. However, fermions have spin $\mathbb{Z}/2$, thus for a 2π rotation of space, $|\psi\rangle \mapsto -1\,|\psi\rangle$. In most cases, these are the only spin values possible. However, there is one crucial case where particles are allowed to take on other spin values, which we will discuss in the next section.

A. Spin in 2 vs 3 Dimensions

In 1, 3, and 3+ dimensions, results of spin statistics prove that the only possible spins are integer and half-integer, as discussed in the previous section. In this section, we hope to answer the following questions: what happens in two dimensions, and why does it allow for more spin?

To answer this question, we consider the Lie groups $SO(2)$ and $SO(3)$, which encode the rotational symmetries in 2 and 3 dimensions respectively. We can understand what makes phase-reactions to symmetries distinct by studying the symmetries themselves. First, let's investigate differences between these two groups [4]:

	$SO(2)$	$SO(3)$
Identified with	S^1	$\mathbb{RP}^3$
Fundamental group	$\mathbb{Z}$	$\mathbb{Z}_2$
Nullhomotopic paths	trivial	$p+p$ for all paths

As Lie groups, both $SO(2)$ and $SO(3)$ are manifolds.

We can examine their topological equivalence, called "homeomorphism", to other manifolds we are more familiar with[1]. (This is similar to homotopy equivalence, but stricter.) We can say $SO(2) \cong S^1$, the real unit circle, in the sense that every element of $SO(2)$ is a 2D rotation, which can be represented on a unit circle [5]. One may think this is easily generalized to increasingly higher-dimensional spheres, however this is not the case. In fact, $SO(3) \cong \mathbb{RP}^3$, the real projective plane in three dimensions (and, secretly, $S^1 \cong \mathbb{RP}^2$). This is a manifold of all lines through the origin. The proof of this homeomorphism is beyond the scope of this paper, however intuition follows from the fact that all three dimensional rotations are identified by an axis and an angle. If one stores the rotation as the radial distance from the origin along the axis, this generates a solid ball of radius π. It follows from basic geometry that a rotation of π around a radius r is exactly π around $-r$, the opposite radius, which means antipodal points are identified with each other. The following figure illustrates these manifolds.

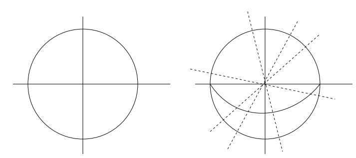


Fig. 3. Left: S^1, Right: $\mathbb{RP}^3$

Now we discuss a second property of these groups, called the fundamental group. The fundamental group of a manifold is a tool used in algebraic topology to study the number and types of homotopy-equivalent loops on a manifold.

From algebraic topology, we know that the fundamental group of S^1 is $\mathbb{Z}$, ie that $\pi_1(S_1) \cong \mathbb{Z}$ [4]. The full algebraic topology discussion is also beyond the scope of this paper, however, we may discussion the intuition here. A fundamental group of $\mathbb{Z}$ essentially means that any loop done can only be removed by a loop in the opposite direction [6]. This means that any two loops are fundamentally different from each other, and in fact any partial traversal of the circle does not have any homotopically-equivalent other path. Thus each loop or partial loop is part of its own equivalence class and any rotation of space is fundamentally different.

[1] Some details are obscured here; these are actually *abelianizations*.

Similarly, we know that $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$ [4]. Again we will only discuss the intuition of this fact. It means that every path on the manifold is in exactly one of two equivalence classes: **1** or **-1**. If a path is in **1**, this means it is trivial and can be shrunk to a point. If a path is in **-1**, there is something critically different about it that does not allow it to be shrunk to a point; however, if the path is traversed twice, the combined traversal cancels out the **-1** $\times$ **-1** = **1**. Thus traversing any path twice results in a path that can be shrunk to a point. This exactly corresponds to the third row in the table: **nullhomotopic paths** are simply paths which can be shrunk to a point. Crucially the only eventually-nullhomotopic paths in $SO(2)$ are the the paths which map to finite-order elements in $\mathbb{Z}$: of which exactly only the trivial paths do. Similarly, all elements in $\mathbb{Z}_2$ are finite-order, thus all paths are eventually nullhomotopic if traversed enough times [7].

It should be clear how the manifolds, their fundamental groups, and nullhomotopic paths inform spin in different dimensions. In all dimensions but 2, all paths either are nullhomotopic or require one extra traversal: this corresponds to the acquired phase of -1 in fermions. However, in 2 dimensions the story is different: all rotations of space are unique, and thus particles of all spin are allowed.

Thus in 2D we can work with particles of any spin– **anyons**, which are required for topological quantum computing [1].

IV. The Braid Group B_n

Now that we have discussed how particles can acquire phase from rotations around each other and how those different phases can be dictated, we turn to a larger system: how do we calculate the interactions of multiple particles swapping with each other?

In most dimensions, permuting n particles leads to the permutation group S_n, because generated phase is either -1 or 1 and uninteresting. However, in 2 dimensions as we have discovered, phase is more interesting depending on the spin of your anyons. Thus we must be able to record the order in which we swap anyons [1].

Consider three anyons **A**, **B**, and **C** embedded in a two-dimensional plane, with the arrow of time shown on the right in the figure below.

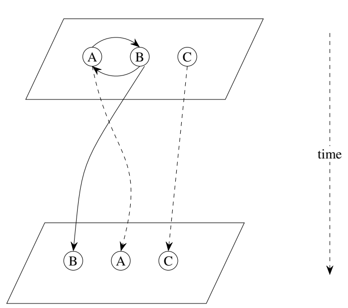


Fig. 4. The swap σ_1.

When you swap anyons **A** and **B**, viewed over the course of time, the worldlines of the anyons form a path. Importantly, these worldlines do not intersect, and the specifics of these paths are unimportant: only the other strands which they are intertwined with, and in what order, matter. Thus these paths are equivalent under homotopy. This type of topological intertwined path of n strands is called an **n-strand braid.** The behavior of these paths is encoded in the braid group B_n, which is defined as follows.

Definition IV.1 (The Braid Group B_n)**.** The braid group on n strands is the free group on $n-1$ generators [1], subject to restrictions R_1 and R_2:

$$B_n = \{\sigma_1, ..., \sigma_{n-1} | R_1, R_2\}$$

Where σ_i is a swap of particles p_i and p_{i+1}.

- R_1: Disjoint swaps commute.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2$$

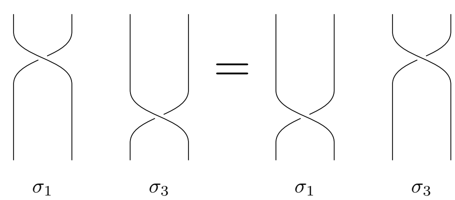
- R_2: Yang-Baxter braid relation.

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \forall \ i \in 1, ..., n-2$$

We will demonstrate both relations in order to motivate why they are added to the group.

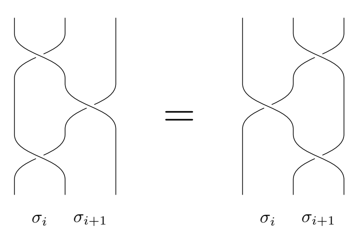
A. R_1

First we examine the first relation, which shows that disjoint swaps commute. Take the following example, where order is depicted by time, top to bottom. On four strands, σ_1 swaps p_1, p_2, where σ_3 swaps p_3, p_4. Regardless of their order, this results in the same topological braid as seen below.



B. R_2

Next we consider the braid relation, which is depicted in the following figure. While the two braids may appear to be different, they are topologically equivalent.



V. Future Work

In the next paper and presentation, I will discuss the following topics:

- Encoding qubits and gates
- Fusion and measurement
- How universality is achieved

REFERENCES

[1] J. Preskill, "Lecture notes for physics 229, chapter 9: Topological quantum computation." https://www.preskill.caltech.edu/ph229/notes/chap9.pdf, 2004.

[2] V. Kaplunovsky, "Aharonov-bohm effect." Lecture notes for 2024F-QFT, University of Texas at Austin, 2024.

[3] F. Wilczek, "Quantum mechanics of fractional-spin particles," *Physical Review Letters*, vol. 49, pp. 957–959, Oct 1982.

[4] A. Hatcher, *Algebraic Topology*, ch. 1.1: The Fundamental Group of the Circle. Cambridge University Press, 2001.

[5] L. W. Tu, *An Introduction to Manifolds*. Universitext, New York: Springer, 2nd ed., 2011.

[6] C.-H. P. Lee, "Mathematical foundations of topological quantum computation," reu paper, University of Chicago, Department of Mathematics, 2016. Mentors: Peter May and various graduate students.

[7] C.-H. P. Lee, "Introduction to topological quantum computation with anyons," reu paper, University of Chicago, Department of Mathematics, 2018. Mentors: Peter May and various graduate students.